Ideals of Pre $A^*$-Algebra

A. Satyanarayana and J. Venkateswara Rao

Abstract—In this paper we formulate the definition of an ideal of $A^*$-algebra $A$ and discuss certain examples. Also certain binary operations are introduced on the set of ideals and various properties of these are investigated. Copyright © 2011 Yang’s Scientific Research Institute, LLC. All rights reserved.

Index Terms—Pre $A^*$-algebra, ideal, principal ideal.

I. INTRODUCTION

In 1948 the study of lattice theory had been made by Birkhoff [1]. In a draft paper [3], The Equational Theory of Disjoint Alternatives around 1989, E. G. Maines introduced the concept of Ada (Algebra of disjoint alternatives) $\langle A, \land, \lor, (-)^\land, (-)^\lor, 0, 1, 2 \rangle$ which is however differ from the definition of the Ada of his later paper [3] As and the equational theory of if-then-else in 1993. While the Ada of the earlier draft seems to be based on extending the If-Then-Else concept more on the basis of Boolean Algebra and the later concept is based on C-algebra $(A, \land, \lor, \prime)$ introduced by Fernando Guzman and Craig. C. Squir [2].

In 1994, P. Koteswara Rao [7] first introduced the concept $A^*$-Algebra $(A, \land, \lor, \prime, (-)^\prime)$ not only studied the equivalence with Ada, C-algebra, Adas and the connection with 3-Ring, Stone type representation but also introduced the concept of $A^*$-clone, the If-Then-Else concept over $A^*$-algebra and Ideal of $A^*$-algebra. In 2000, J. Venkateswara Rao [4] introduced the concept of Pre $A^*$-algebra $(A, \land, \lor, (-)^\land)$ as the variety generated by the 3-element algebra $A = \{0, 1, 2\}$ which is an algebraic form of three valued conditional logic. It was proved that the only sub directly irreducible Pre $A^*$-algebra are $A$ or two element Boolean algebra $B = \{0, 1\}$. In [6] Rao et al. generated Pre $A^*$-algebras from Boolean algebras and defined congruence relation and Ternary operation on $A$. In [5], J. Venkateswara Rao and K. Srinivasa Rao defined a partial ordering on a Pre $A^*$-algebra $A$ and the properties of $A$ as a poset are studied.

In this paper we formulate the definition of an ideal of Pre $A^*$-algebra $A$ and discuss certain examples. Also certain binary operations are introduced on the set of ideals and various properties of these are investigated.

II. PRELIMINARIES

Definition II.1. Boolean algebra is an algebra $(B, \lor, \land, (-)^\land, 0, 1)$ with two binary operations, one

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Note II.1. The elements 0, 1, 2 in the above example satisfy the following laws:

(a) $2^\land = 2$;
(b) $1 \lor x = x$ for all $x \in 3$;
(c) $0 \land x = x$ for all $x \in 3$;
(d) $2 \lor x = 2 \lor x = 2$ for all $x \in 3$.

Example II.2. $2 = \{0, 1\}$ with operations $\land, \lor, (-)^\land$ defined below is a Pre $A^*$-algebra.

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Note II.2. (i) $(2, \lor, \land, (-)^\land)$ is a Boolean algebra. So every Boolean algebra is a Pre $A^*$-algebra.

(ii) The identities II.2(a) and II.2(d) imply that the varieties of Pre $A^*$-algebras satisfies all the dual statements of II.2(b) to II.2(g).
Note II.3. Let $A$ be a Pre $A^*$-algebra then $A$ is Boolean algebra iff $x \lor (x \land y) = x$, $x \land (x \lor y) = x$ (absorption laws holds).

**Lemma II.1.** [5] Every Pre $A^*$-algebra satisfies the following laws.

(a) $x \lor (x^\sim \land x) = x$;
(b) $x \lor (x^\sim \land y) = (x \land y) \lor (x^\sim \land y)$;
(c) $(x \lor x^\sim) \land x = x$;
(d) $(x \lor y) \land z = (x \land z) \lor (x^\sim \land y \land z)$.

**Definition II.3.** [5] Let $A$ be a Pre $A^*$-algebra. An element $x \in A$ is called central element of $A$ if $x \lor x^\sim = 1$ and the set $\{x \in A : x \lor x^\sim = 1\}$ of all central elements of $A$ is called the centre of $A$ and it is denoted by $B(A)$. Note that if $A$ is a Pre $A^*$-algebra with 1 then $1, \emptyset \in B(A)$. If the centre of Pre $A^*$-algebra coincides with $\{0, 1\}$ then we say that $A$ has trivial centre.

**Theorem II.1.** [5] Let $A$ be a Pre $A^*$-algebra with 1, then $B(A)$ is a Boolean algebra with the induced operations $\land$, $\lor$, $(\sim)^\sim$.

**Lemma II.2.** [5] Let $A$ be a Pre $A^*$-algebra with 1,

(a) If $y \in B(A)$ then $x \land x^\sim \land y = x \land x^\sim$, $\forall x \in A$;
(b) $x \land (x \lor y) = x \lor (x \land y) = x$ if and only if $x, y \in B(A)$.

### III. Ideals of Pre $A^*$-algebra

**Definition III.1.** A nonempty subset $U$ of a Pre $A^*$-algebra $A$ is said to be an ideal of $A$ if the following hold

(i) $a, b \in U \Rightarrow a \lor b \in U$;
(ii) $a \in U \Rightarrow x \land a \in U$ for each $x \in A$.

**Example III.1.** All the ideals of Pre $A^*$-algebra $A = \{0, 1, 2\}$ are $I_1 = \{2\}, I_2 = \{0, 2\}$ and $A$ itself.

Now we give some examples of Pre $A^*$-algebras and collect all the ideals of these.

**Example III.2.** Let $G = \{a_1, a_2, a_3, a_4, a_5\}$ where $a_1 = (1, 2)$, $a_2 = (0, 2)$, $a_3 = (2, 1)$, $a_4 = (2, 0)$, $a_5 = (2, 2)$. Then $G$ is a Pre $A^*$-algebra (a sub algebra of $A \times A$) under the point wise operations given in the following tables:

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This algebra $(G, \lor, \land, (\sim)^\sim)$ is a Pre $A^*$-algebra without 1. All the ideals of $G$ are:

- $I_1 = \{a_3\}$,
- $I_2 = \{a_2, a_5\}$,
- $I_3 = \{a_1, a_5\}$,
- $I_4 = \{a_1, a_2, a_5\}$,
- $I_5 = \{a_2, a_4, a_5\}$,
- $I_6 = \{a_3, a_4, a_5\}$,
- $I_7 = \{a_1, a_2, a_4, a_5\}$,
- $I_8 = \{a_2, a_3, a_4, a_5\}$,
- $I_9 = G$.

**Example III.3.** Let $H = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$ where $b_1 = (1, 2)$, $b_2 = (0, 2)$, $b_3 = (2, 1)$, $b_4 = (2, 0)$, $b_5 = (2, 2)$, $b_6 = (1, 1)$, $b_7 = (0, 0)$. Then $H$ is a Pre $A^*$-algebra (a sub algebra of $A \times A$) under the point wise operations given in the following tables:

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This algebra \((H, \land, \lor, (-)^\sim)\) is a Pre \(A^*\)-algebra without 1 and 1 = \(b_6\). All the ideals of \(H\) are

\[
\begin{align*}
I_1 &= \{b_5\}, \\
I_2 &= \{b_2, b_5\}, \\
I_3 &= \{b_4, b_5\}, \\
I_4 &= \{b_1, b_2, b_5\}, \\
I_5 &= \{b_3, b_4, b_5\}, \\
I_6 &= \{b_1, b_2, b_4, b_5\}, \\
I_7 &= \{b_2, b_3, b_4, b_5\}, \\
I_8 &= \{b_2, b_4, b_5, b_7\}, \\
I_9 &= \{b_1, b_2, b_3, b_4, b_5\}, \\
I_{10} &= \{b_1, b_2, b_4, b_5, b_7\}, \\
I_{11} &= \{b_1, b_2, b_3, b_4, b_5, b_7\}, \\
I_{12} &= \{b_2, b_3, b_4, b_5, b_7\}, \\
I_{13} &= H.
\end{align*}
\]

**Example III.4.** Let \(F = A \times A = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9\}\) where \(c_1 = (1, 2), c_2 = (0, 2), c_3 = (2, 1), c_4 = (2, 0), c_5 = (2, 2), c_6 = (1, 1), c_7 = (0, 0), c_8 = (1, 0), c_9 = (0, 1)\). Then \(F\) is a Pre \(A^*\)-algebra under the point wise operations given in the following tables.

This algebra \((F, \land, \lor, (-)^\sim)\) is a Pre \(A^*\)-algebra with 1 and 1 = \(c_6\). The ideals of \(A \times A\) need not be of the form \(I_1 \times I_2\) where \(I_1, I_2\) are ideals of \(A\) for such ideals of \(A \times A\) are 9 in number (since \(C\) has only 3 ideals namely \(\{2\}, \{0, 2\}\), and \(A\)), where we are exhibiting 14 ideals in \(A \times A\).

All the ideals of \(F\) are

\[
\begin{align*}
I_1 &= \{c_3\}, \\
I_2 &= \{c_2, c_5\}, \\
I_3 &= \{c_4, c_5\}, \\
I_4 &= \{c_1, c_2, c_5\}, \\
I_5 &= \{c_3, c_4, c_5\}, \\
I_6 &= \{c_1, c_2, c_4, c_5\}, \\
I_7 &= \{c_2, c_3, c_4, c_5\}, \\
I_8 &= \{c_2, c_4, c_5, c_7\}, \\
I_9 &= \{c_1, c_2, c_3, c_4, c_5\}, \\
I_{10} &= \{c_1, c_2, c_4, c_5, c_7\}, \\
I_{11} &= \{c_1, c_2, c_3, c_4, c_5, c_7\}, \\
I_{12} &= \{c_1, c_2, c_4, c_5, c_7, c_8\}, \\
I_{13} &= \{c_2, c_3, c_4, c_5, c_7, c_9\}, \\
I_{14} &= F.
\end{align*}
\]

**Lemma III.1.** Let \(A\) be a Pre \(A^*\)-algebra and \(p \in A\). Then \(\{x \land p/x \in A\}\) is the smallest ideal containing \(p\).

**Proof.** Let \(U = \{x \land p/x \in A\}\).

If \(s, t \in U\) then \(s = x \land p\) and \(t = y \land p\) for some \(x, y \in A\).

\[
\begin{align*}
(s \lor t) &= (x \land p) \lor (y \land p) \\
&= (x \land p) \lor (y \land p) \\
&= [(x \land p) \lor (y \land p)] \lor (p \land p) \\
&= (s \lor t) \land p \in U.
\end{align*}
\]

Therefore \(s \lor t \in U\).
Lemma III.2. Let A be a Pre $A^*$-algebra and p ∈ A then \{x ∧ p/x ∈ A\} is called the Principal ideal generated by p and is denoted by $\langle p \rangle$.

Now we give some examples of principal ideals of certain Pre $A^*$-algebras.

Example III.5. Principal ideals of the Pre $A^*$-algebra $A = \{0, 1, 2\}$

\begin{align*}
\langle 0 \rangle &= \{0, 2\}, \\
\langle 1 \rangle &= \{0, 1, 2\}, \\
\langle 2 \rangle &= \{2\}.
\end{align*}

Example III.6. Principal ideals of the Pre $A^*$-algebra $G = \{a_1, a_2, a_3, a_4, a_5\}$ (see Example III.2)

\begin{align*}
\langle a_1 \rangle &= \{a_1, a_2, a_5\}, \\
\langle a_2 \rangle &= \{a_2, a_5\}, \\
\langle a_3 \rangle &= \{a_3, a_4, a_5\}, \\
\langle a_4 \rangle &= \{a_4, a_5\}, \\
\langle a_5 \rangle &= \{a_5\}.
\end{align*}

Example III.7. Principal ideals of the Pre $A^*$-algebra $H = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$ (see Example III.3)

\begin{align*}
\langle b_1 \rangle &= \{b_1, b_2, b_5\}, \\
\langle b_2 \rangle &= \{b_2, b_5\}, \\
\langle b_3 \rangle &= \{b_3, b_4, b_5\}, \\
\langle b_4 \rangle &= \{b_4, b_5\}, \\
\langle b_5 \rangle &= \{b_5\}, \\
\langle b_6 \rangle &= H, \\
\langle b_7 \rangle &= \{b_2, b_4, b_5, b_7\}.
\end{align*}

Example III.8. Principal ideals of the Pre $A^*$-algebra $F = \{b_1, b_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9\}$ (see Example III.4)

\begin{align*}
\langle c_1 \rangle &= \{c_1, c_2, c_5\}, \\
\langle c_2 \rangle &= \{c_2, c_5\}, \\
\langle c_3 \rangle &= \{c_3, c_4, c_5\}, \\
\langle c_4 \rangle &= \{c_4, c_5\}, \\
\langle c_5 \rangle &= \{c_5\}, \\
\langle c_6 \rangle &= F, \\
\langle c_7 \rangle &= \{c_2, c_4, c_5, c_7\}, \\
\langle c_8 \rangle &= \{c_1, c_2, c_4, c_5, c_7, c_8\}, \\
\langle c_9 \rangle &= \{c_2, c_3, c_4, c_5, c_7, c_9\}.
\end{align*}

Lemma III.2. Let A be a Pre $A^*$-algebra and p, y ∈ A then \( y ∈ \langle p \rangle \Rightarrow y = y ∧ p \).

Proof. Suppose \( y ∈ \langle p \rangle \) ⇒ \( y = x ∧ p \) for some \( x ∈ A \). Now

\begin{align*}
y ∧ p &= (x ∧ p) ∧ p \\
&= x ∧ p \\
&= y.
\end{align*}

Conversely suppose that \( y = y ∧ p \). Then \( y ∧ p ∈ \langle p \rangle \). Therefore \( y ∈ \langle p \rangle \).

Lemma III.3. Let A be a Pre $A^*$-algebra and p, q ∈ A then \( \langle p \rangle \cap \langle q \rangle = \langle p ∧ q \rangle \).

Proof. Let p, q ∈ A.

Let \( x ∈ \langle p ∧ q \rangle \) ⇒ \( x = x ∧ p ∧ q \)

\begin{align*}
x ∧ p &= x ∧ p ∧ q \quad \Rightarrow \quad x ∈ \langle q \rangle \\
&⇒ \quad x ∈ \langle p ∧ q \rangle \\
&⇒ \quad x ∈ \langle p \rangle \cap \langle q \rangle
\end{align*}

Therefore \( \langle p ∧ q \rangle ⊆ \langle p \rangle \cap \langle q \rangle \).

Let \( x ∈ \langle p \rangle \cap \langle q \rangle \). Then \( x ∈ \langle p \rangle \) and \( x ∈ \langle q \rangle \).

\begin{align*}
x &= x ∧ p \quad \Longrightarrow \quad x = x ∧ q \\
&⇒ \quad x ∈ \langle p ∧ q \rangle \\
&⇒ \quad x ∈ \langle p \rangle \cap \langle q \rangle
\end{align*}

Therefore \( \langle p ∧ q \rangle ⊆ \langle p \rangle \cap \langle q \rangle \).

Thus \( \langle p \rangle \cap \langle q \rangle = \langle p ∧ q \rangle \).

Note III.1. We do not have \( \langle p \rangle \cup \langle q \rangle = \langle p ∨ q \rangle \), in general for example in the Pre $A^*$-algebra $G = \{a_1, a_2, a_3, a_4, a_5\}$ \( \{a_1\} \cup \{a_2, a_5\} \cup \{a_3, a_4, a_5\} = G \). But \( a_1 ∨ a_3 = a_5 \) and \( a_5 \neq G \).

Theorem III.1. Let p and q are two elements of a Pre $A^*$-algebra A then \( \langle p ∧ q \rangle ⊆ \langle p \rangle \).

Proof. Let \( x ∈ \langle p ∧ q \rangle \) ⇒ \( x = x ∧ p ∧ q \). Then \( x ∧ p = x ∧ p ∧ q ∧ p = x ∧ p ∧ q = x = x ∈ \langle p \rangle \). Therefore \( \langle p ∧ q \rangle ⊆ \langle p \rangle \).

Moreover we have \( \langle p ∧ q \rangle ⊆ \langle q \rangle \).
Lemma III.5. If $p$ and $q$ are two elements of a Pre $A^*$-algebra $A$ then $\langle p \rangle \lor \langle q \rangle = \{p, q\}$.

Proof. Let $\langle \{p, q\} \rangle = U$. Clearly $\langle p \rangle \subseteq U$, $\langle q \rangle \subseteq U$ and hence $\langle p \rangle \lor \langle q \rangle \subseteq U$. Conversely the fact that $\{p, q\} \subseteq \langle p \rangle \lor \langle q \rangle$. Therefore $\langle p \rangle \lor \langle q \rangle = \{p, q\}$. □

Theorem III.2. Let $A$ be a Pre $A^*$-algebra and $p, q \in A$. If $\langle p \rangle = \langle q \rangle$ then $\langle s \lor p \rangle = \langle s \lor q \rangle$.

Proof. Suppose $\langle p \rangle = \langle q \rangle$ then $p = q$. Now $(s \lor p) \land (s \lor q) = s \lor (p \land q) = s \lor p$.

$$(s \lor p) \land (s \lor q) = s \lor (p \land q) = s \lor q$$

$$(s \lor p) \land (s \lor q) = s \lor (p \land q) = s \lor q$$

Therefore $\langle s \lor p \rangle = \langle s \lor q \rangle$. □

Definition III.3. For any subset $S$ of a Pre $A^*$-algebra $A$ and an ideal $U$, we define $(S : U) = \{ a \in A / s \land a \in U \text{ for each } s \in S \}$. If $S = \{c\}$ then we write it as $(c : U)$.

Theorem III.3. $(S : U)$ is an ideal of $A$.

Proof. Let $a, b \in (S : U)$. Then $a \land b \in U$ for all $s \in S$, $a \land (a \lor b) = (a \land a) \lor (a \land b) \in U$ (since $s \land a$, $s \land b \in U$ and $U$ is ideal). Hence $a \land b \in (S : U)$. Let $a \in (S : U)$, $x \in A \Rightarrow a \land s \in U$ for some $s \in S$. Then $(x \land a) \land s = x \land (a \land s) \in U$ (since $a \land s \in U$, $x \in A$ and $U$ is ideal). Therefore $x \land a \in (S : U)$. Thus $(S : U)$ is an ideal of $A$. □

Theorem III.4. Let $S$ be a subset and $U$ an ideal of a Pre $A^*$-algebra $A$ then $(S : U) = A \iff S \subseteq U$.

Proof. Suppose $(S : U) = A$. Then $s \in S \Rightarrow s \in A = (S : U)$ (since $S \subseteq A \Rightarrow s \land a \in U \Rightarrow s \in U$. Therefore $S \subseteq U$.

Conversely suppose that $S \subseteq U$. Since $(S : U)$ is an Ideal of $A$, $(S : U) \subseteq A$. Let $x \in S$. Since $U$ is an ideal $x \land s \in U$ for each $s \in U$. Therefore $x \land s \in U$ for each $s \in S$ (since $S \subseteq U$). Thus $x \in (S : U) \Rightarrow A \subseteq (S : U)$. Therefore $(S : U) = A$. □

Theorem III.5. For any ideal $U$ of $A$ and $S \subseteq A$ then $U \subseteq (S : U)$.

Proof. Let $a \in U$. Since $U$ is an ideal $a \land s \in U$ for all $s \in S$. Therefore $a \land s \in U$ for all $s \in S$. Thus $a \in (S : U)$. Therefore $U \subseteq (S : U)$. □

Theorem III.6. Let $S$ be a subset and $U$ and $W$ are the two ideals of Pre $A^*$-algebra $A$ then $(S : U \cap W) = (S : U) \cap (S : W)$.

Proof. Let $x \in A$. Then $x \in (S : U \cap W) \iff x \land s \in U \cap W$, for all $s \in S$.

$$(x \land s) \in U \land (x \land s) \in W \text{ for all } s \in S$$

$$(x \land s) \in U \land (x \land s) \in W \text{ for all } s \in S$$

Therefore $(S : U \cap W) = (S : U) \cap (S : W)$. □

Theorem III.7. Let $S$ and $T$ be any two subsets of Pre $A^*$-algebra $A$ and $S \subseteq T$ then $(T : U) \subseteq (S : U)$.

Proof. Let $x \in (T : U)$ then $x \land s \in U$ for all $t \in T$ and hence $x \land s \in U$ for all $s \in S$ (since $S \subseteq T$). So that $x \in (S : U)$. Therefore $(T : U) \subseteq (S : U)$. □

Theorem III.8. Let $S$ and $T$ be any two subsets of Pre $A^*$-algebra $A$ then $(S \cup T : U) = (S : U) \cap (T : U)$.

Proof. We know that $S$ and $T \subseteq S \cup T$ and hence by above lemma $(S \cup T : U) \subseteq (S : U) \cap (T : U)$. Now let $x \in (S \cup T : U) \cap (T : U)$. Then $x \land s \in U$ and $x \land t \in U$ for all $s \in S$ and $t \in T$. Therefore $x \land a \in U$ for all $a \in S \cup T$. $x \in (S \cup T : U)$. Therefore $x \in (S : U) \cap (T : U)$. Thus $(S \cup T : U) = (S : U) \cap (T : U)$. □

Theorem III.9. Let $A$ be a Pre $A^*$-algebra $S \subseteq A$ for any ideals $U$ and $W$ of $A$. $(S) \cap W \subseteq W \subseteq (S : U)$.

Proof. Suppose that $W \subseteq (S : U)$. Let $x \in (S) \cap W$. Then $x = \bigwedge_{i=1}^{n}(y_i \land s_i)$ and $x \in (S : U)$ where $s_i \in S$, $y_i \in A$. Therefore $x \land s_i \in W$ for each $i$. Now

$$x = x \land x$$

$$= x \land \bigwedge_{i=1}^{n}(y_i \land s_i)$$

$$= \bigwedge_{i=1}^{n}(x \land y_i) \land (x \land s_i)$$

Therefore $x \in U$ (since $x \land s_i \in U$). Therefore $(S) \cap W \subseteq U$ and hence $(S) \cap W \subseteq U$ (since $W \subseteq (S : U)$). On the other hand $W$ is an ideal of $A$ such that $(S) \cap W \subseteq U$. Let $x \in W$. Then for any $s \in S$, $x \land s \in (S) \cap W$ (since $W$ and $(S)$ are ideals) $\Rightarrow x \land s \in U$ (by supposition). Therefore $x \in (S : U)$. Thus $W \subseteq (S : U)$. □


Proof. We know that $S \subseteq (S : U)$ and therefore by Theorem III.8 $(S : U) \subseteq (S : U)$. Let $x \in (S : U)$. Then $x = x \land s \in U$ for all $s \in S$. Let $t \in (S)$. Then $t = \bigwedge_{i=1}^{n}(y_i \land s_i)$, for some $s_i \in S$, $y_i \in A$. Now

$$x = x \land x$$

$$= x \land \bigwedge_{i=1}^{n}(y_i \land s_i)$$

$$= \bigwedge_{i=1}^{n}(x \land y_i) \land (x \land s_i)$$

Then $x \land t \in U$. Thus $x \in (S : U)$. Therefore $(S : U) \subseteq (S : U)$. Hence $(S : U) = (S : U)$. □

Theorem III.10. Let $I, J$ and $K$ be ideals of a Pre $A^*$-algebra $A$ then $(I \lor J : K) = (I : K) \cap (J : K)$.

Proof. $(I \lor J : K) = ((I \cup J) : K)$

$$= (I \cup J : K) \text{ (by Lemma III.6)}$$

$$= (I : K) \cap (J : K) \text{ (by Theorem III.8)}$$
REFERENCES